Parityizing Rabin and Streett*

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Abstract

The parity acceptance condition for ω-regular languages is a special case of the Rabin and Streett acceptance conditions. While the parity acceptance condition is as expressive as the richer conditions, in both the deterministic and nondeterministic settings, Rabin and Streett automata are more succinct, and their translation to parity automata may blow-up the state space. The appealing properties of the parity condition, mainly the fact it is dualizable and allows for memoryless strategies, make such a translation useful in various decision procedures.

In this paper we study languages that are recognizable by an automaton on top of which one can define both a Rabin and a Streett condition for the language. We show that if the underlying automaton is deterministic, then we can define on top of it also a parity condition for the language. We also show that this relation does not hold in the nondeterministic setting. Finally, we use the construction of the parity condition in the deterministic case in order to solve the problem of deciding whether a given Rabin or Streett automaton has an equivalent parity automaton on the same structure, and show that it is PTIME-complete in the deterministic setting and is PSPACE-complete in the nondeterministic setting.

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1 Introduction

Finite *automata on infinite objects* are widely used for the specification, verification, and synthesis of nonterminating systems [3, 15, 21]. Since a run of an automaton on an infinite word does not have a final state, acceptance is determined with respect to the set of states visited infinitely often during the run. There are many ways to classify an automaton on infinite words. One is the class of its acceptance condition. For example, in *Büchi* automata, some of the states are designated as accepting states, and a run is accepting iff it visits states from the accepting set infinitely often [1]. More general are *Rabin* automata. Here, the acceptance condition is a set $\alpha = \{\langle G_1, B_1\rangle, \ldots, \langle G_k, B_k\rangle\}$ of pairs of sets of states, and a run is accepting if there is a pair $\langle G_i, B_i\rangle$ for which the set of states visited infinitely often intersects G_i and does not intersect B_i. The condition α can also be viewed as a *Streett* condition, in which case a run is accepting if for all pairs $\langle G_i, B_i\rangle$, if the set of states visited infinitely often intersects G_i, then it also intersects B_i. Note that the Rabin and Streett conditions dualize each other. Thus, a run satisfies α when viewed as a Rabin condition iff it does not satisfy α when viewed as a Streett condition. The analysis of logics with fixed-points led to extensive study of the *parity* acceptance condition [5, 17]. There, the acceptance condition is a sequence $\{F_1, F_2, \ldots, F_{2k}\}$ of sets of states, and a run is accepting iff the minimal index i for which the set F_i is visited infinitely often is even. It is not hard to see that the parity condition is a special case of both the Rabin and Streett conditions, in the sense that a given parity condition can be translated to equivalent Rabin and Streett conditions. The number of pairs or sets in the acceptance conditions is referred to as the

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index of the automaton. We use NRW, NSW, and NPW to denote nondeterministic Rabin, Streett, and parity word automata, respectively, and use DRW, DSW, and DPW to denote the corresponding deterministic automata. We sometimes add the number of states and index. So, for example, NRW(n, k) is a nondeterministic automaton with n states and index k.

The type of an automaton influences its succinctness. For example, while Rabin, Streett, and parity automata all recognize all ω-regular languages, the translation of a DRW to a DSW (or vise versa) may involve a blow-up exponential in the index, and so does the translation of a DRW or a DSW to a DPW [16]. The succinctness of Rabin and Streett automata with respect to parity automata is carried over to the nondeterministic setting [19, 20]. The type of an automaton also influences the difficulty of constructions and decision problems for it. For example, while complementation of DPWs is straightforward, as it is easy to dualize a parity condition, complementation of DRWs and DSWs involves a translation of the dual DSWs and DRWs, respectively, back to DRWs and DSWs, which, as described above, involves an exponential blow-up. As another example, while the nonemptiness problem for DPW(n, k) can be solved in time $O(n \log k)$ [10] and is NLOGSPACE-complete, the one for DRW(n, k) is still NLOGSPACE-complete but needs time $O(nk)$, whereas the one for DSW(n, k) is PTIME-complete [6], with on-going research on the precise, larger than $O(nk)$, bound [6, 8]. Thus, the succinctness of Rabin and Streett automata is traded-off by more complex constructions and algorithms. Finally, only the parity acceptance condition allows for memoryless strategies for both players [5]. The fact parity games are memoryless is of great importance in synthesis algorithms, where one wants to generate transducers for the winning strategies [4]. The fact both players have memoryless strategies is useful in settings in which one considers strategies for both the system and its environment [13]. A good evidence to the superiority of the parity condition in the application front is the fact that the highlight of Piterman's determinization construction for nondeterministic Büchi automata [18] has been the fact it generates a DPW, rather than the DRW generated by Safra's construction [19], and less the saving in the state space it suggests.

Recall that while the parity condition can be translated to the Rabin and Streett conditions, the other direction is not valid: the translations of Rabin and Streett automata to parity automata cannot only modify the acceptance condition and they involve automata with different, and substantially bigger, state spaces. In some cases, it is possible to translate automata with a particular acceptance condition to automata with a weaker acceptance condition without modifying the state space. For example, it is shown in [11] that DRWs are *Büchi type*: if a DRW has an equivalent deterministic Büchi automaton, then there is also an equivalent deterministic Büchi automaton on top of the same structure. Additional examples of typeness for ω-regular languages are studied in [12]. We would like to study typeness for parity automata, and in particular the ability to modify Rabin and Streett conditions to an equivalent parity condition.

The connection between the combination of Rabin and Streett with the parity condition was studied in the context of *two-player games* in [22]. There, Zielonka shows that if the winning condition of a finitely colored game can be specified as both Rabin and Streett conditions, then it can also be characterized by a parity (or chain, as it is called there) condition. In this paper we study this connection in the context of automata on infinite words: Suppose that some language can be defined on top of the same automaton by both Rabin and Streett conditions. Can we define an equivalent parity condition on top of the same automaton? Before describing our results, let us mention that they do not follow directly from Zielonka's result. In fact, our results are part of a general effort of lifting results from the world of two-player games to the world of automata. By [7], the nonemptiness problem for nondeterministic tree automata can be reduced to the solution of a two-player game. The connection between games and automata is further formalized in [14]. As shown there, since transitions of games are not associated with letters, games correspond to alternating word automata over a singleton alphabet, and one cannot talk about a language of a game. Indeed, results and methods that hold for games cannot in general be applied to automata. For example, while today there are several algorithms that solve parity games in time less than $O(n^k)$ [9], the best translation of alternating parity word automata to alternating weak word automata (for which the 1-letter nonemptiness problem can be solved in linear time) involves an $O(n^k)$ blow up, where n is the number of states and k is the index of the parity condition. The challenge has to do with the fact that reasoning about games one can abstract components of the game, whereas translations among automata must keep the exact same language – every letter counts.

Back to our problem, our main result states that if the automaton is deterministic, then one can define an equivalent parity condition on top of it! Formally, if $\mathcal{A}$ is a deterministic automaton with n states and there is a Rabin condition α of index k and a Streett condition β of index l such that the language of $\mathcal{A}$ with α is equal to the language of $\mathcal{A}$ with β, then there exists a parity condition γ of index at most $\min\{2k+2, 2l+2, n+2\}$ such that the language of $\mathcal{A}$ with γ is equal to the language of $\mathcal{A}$ with α and β. Our proof is constructive, it proceeds by induction on the index of the constructed parity automaton, and it involves a decomposition of $\mathcal{A}$ to its maximal strongly connected components, applications of the translation on them, and a composition of the underlying parity conditions to a global one. We study also the nondeterministic setting and show that the determinism of $\mathcal{A}$ is essential. That is, we show that there is a nondeterministic automaton $\mathcal{A}$ such that there are Rabin and Streett conditions on top of $\mathcal{A}$ that define the same language, and still no parity automaton for the language can be defined on top of $\mathcal{A}$. This result is another evidence to the importance of the alphabet and the fact the setting of automata is different than the one of games studied in [22]. Indeed, every nondeterministic automaton can be made deterministic by enriching its alphabet (c.f., the *cylindrification* techniques of [2]).

In addition to formalizing the intuition of "parity is the intersection of Rabin and Streett" and introducing a blow-up-free translation to DPW, a careful analysis of the construction of the equivalent parity condition shows that it is independent of the Streett condition and relies only on its existence. Consequently, we can use the construction in order to decide whether a given DRW can be translated to an equivalent DPW on the same structure. We show that this problem is PTIME-complete. Note that the duality between the Rabin and Streett conditions and the self-duality of the parity condition imply that the problem of deciding whether a given DSW can be translated to a DPW on the same structure is PTIME-complete too. In addition, we prove that the problem of deciding whether a given NRW or NSW has an equivalent NPW on the same structure is PSPACE-complete.

2 Preliminaries

Automata on infinite words. Given an alphabet Σ, an *infinite word over* Σ is an infinite sequence $w = \sigma_0 \cdot \sigma_1 \cdot \sigma_2 \cdots$ of letters in Σ. We denote by w^l the suffix $\sigma_l \cdot \sigma_{l+1} \cdot \sigma_{l+2} \cdots$ of w. An *automaton on infinite words* is $\mathcal{U} = \langle \Sigma, Q, \delta, Q_{in}, \alpha \rangle$, where Σ is the input alphabet, Q is a finite set of states, $\delta : Q \times \Sigma \to 2^Q$ is a transition function, $Q_{in} \subseteq Q$ is a set of initial states, and α is an acceptance condition (a condition that defines a subset of Q^ω).

Since the transition function of $\mathcal{U}$ may specify many possible transitions for each state and letter and since the initial state may be one of the possibly few states in Q_{in}, $\mathcal{U}$ is not *deterministic*. If δ is such that for every $q \in Q$ and $\sigma \in \Sigma$, we have that $|\delta(q, \sigma)| = 1$ and if $|Q_{in}| = 1$, then $\mathcal{U}$ is a deterministic automaton. When $\mathcal{U}$ is deterministic we refer to the single state in Q_{in} by q_{in} and to δ as to a function from Σ^* to Q (rather than to 2^Q). We sometimes refer to the transition function δ of a deterministic automaton as a function $\delta : \Sigma^* \to Q$, where $\delta(\epsilon) = q_{in}$ and $\delta(w \cdot \sigma) = \delta(\delta(w), \sigma))$. Thus $\delta(w)$ is the state that $\mathcal{U}$ visits after reading w. We say that a state $q \in Q$ is reachable in $\mathcal{U}$ if there is a finite word w such that $\delta(w) = q$.

A *run* of $\mathcal{U}$ on w is an infinite word $r = q_0 \cdot q_1 \cdot q_2 \cdots$ over Q, where $q_0 \in Q_{in}$ (i.e., the run starts in an initial state) and for every $l \geq 0$, we have $q_{l+1} \in \delta(q_l, \sigma_l)$ (i.e., the run obeys the transition function). In automata over finite words, acceptance is defined according to the last state visited by the run. When the words are infinite, there is no such thing "last state", and acceptance is defined according to the set $inf(r)$ of states that r visits *infinitely often*, i.e., $inf(r) = \{q \in Q \ : \text{for infinitely many } l \in \mathbb{N}, \text{we have } r_l = q\}$. Hence, acceptance is *prefix independent*, i.e. for all runs r_1, r_2 such that $r_1^l = r_2^m$ for some l and m we have that r_1 is accepting iff r_2 is accepting. As Q is finite, it is guaranteed that $inf(r) \neq \emptyset$. A run r is accepting iff the set $inf(r)$ satisfies the acceptance condition of $\mathcal{U}$. Several acceptance conditions are studied in the literature. We consider here three:

- *Rabin automata*, where $\alpha = \{\langle G_1, B_1\rangle, \langle G_2, B_2\rangle, \ldots, \langle G_k, B_k\rangle\}$, and $inf(r)$ satisfies α iff for some $1 \leq i \leq k$, we have that $inf(r) \cap G_i \neq \emptyset$ and $inf(r) \cap B_i = \emptyset$.
- *Streett automata*, where $\alpha = \{\langle L_1, U_1\rangle, \langle L_2, L_2\rangle, \ldots, \langle L_l, U_l\rangle\}$, and $inf(r)$ satisfies α iff for all $1 \leq i \leq l$, if $inf(r) \cap L_i \neq \emptyset$, then $inf(r) \cap U_i \neq \emptyset$.
- *parity automata*, where $\alpha = \{F_1, F_2, \ldots, F_{2k}\}$ with $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{2k} = Q$, and $inf(r)$ satisfies α iff the minimal index i for which $inf(r) \cap F_i \neq \emptyset$ is even.

The number of sets in the parity acceptance condition or pairs in the Rabin and Streett acceptance conditions is called the *index* of α (or $\mathcal{U}$). Note that the Rabin and Streett conditions are dual, in the sense that a set S satisfies a Rabin condition α iff S does not satisfy α when viewed as a Streett condition. Similarly, the parity condition is dual to itself, in the sense that a set S satisfies a parity condition $\{F_1, F_2, \ldots, F_{2k}\}$ iff S does not satisfy the parity condition $\{\emptyset, F_1, F_2, \ldots, F_{2k}, F_{2k}\}$.

Since $\mathcal{U}$ may not be deterministic, it may have many runs on w. In contrast, a deterministic automaton has a single run on w. An automaton $\mathcal{U}$ is said to accept an input word w iff there exists an accepting run of $\mathcal{U}$ on w. This implies that if $\mathcal{U}$ is deterministic it accepts an input word w iff the single run of $\mathcal{U}$ on w is accepting. The *language* of $\mathcal{U}$, denoted $\mathcal{L}(\mathcal{U})$, is the set of words $\mathcal{U}$ accepts.

A (deterministic) *pre-automaton* $\mathcal{A} = \langle \Sigma, Q, \delta, Q_{in}\rangle$ is a (deterministic, respectively) automaton with no acceptance condition. For an acceptance condition α we use $\mathcal{L}(\mathcal{A}, \alpha)$ to denote the language of the automaton $\mathcal{U} = \langle \mathcal{A}, \alpha\rangle$.

For a pre-automaton $\mathcal{A}$ and a state $q \in Q$, let $\mathcal{A}^q$ denote the pre-automaton $\langle \Sigma, Q, \delta, \{q\}\rangle$. That is, $\mathcal{A}^q$ is the pre-automaton $\mathcal{A}$ except for having q as its single initial state (we sometimes abuse notations and omit the $\{\ \}$ around q). For a pre-automaton $\mathcal{A}$, a subset $C \subseteq Q$ and a state $q \in C$, let $\mathcal{A}|_C^q$ denote the pre-automaton $\langle \Sigma, C, \delta|_C, q\rangle$ where $\delta|_C$ is the restrictions of δ to C, i.e. $\delta|_C : C \times \Sigma \to 2^C$ is such that $\delta|_C(q, \sigma) = \delta(q, \sigma) \cap C$. For an acceptance condition α, denote by $\alpha|_C$ the condition that is obtained from α by intersecting all its sets with C.

The *underlying graph* of a pre-automaton $\mathcal{A}$, denoted $G_{\mathcal{A}}$, is the graph $\langle Q, E\rangle$, where $E(q, q')$ iff there is a letter $\sigma \in \Sigma$ such that $q' \in \delta(q, \sigma)$. A *strongly connected component* of a graph $G = \langle Q, E\rangle$ is a set of vertices $C \subseteq Q$ such that every two states $q, q' \in C$ are reachable from each other. A *maximal strongly connected component* (MSCC) in a graph G is a strongly connected component C such that for all nonempty sets of vertices $C' \in G \setminus C$ the set $C \cup C'$ is not strongly connected. A graph is said to be strongly connected if its vertices consist a single strongly connected component. A pre-automaton is said to be strongly connected if its underlying graph is strongly connected.

2.1 Simple Translations

Syntactic Translations. Parity automata can be viewed as a special case of Rabin and of Streett automata. It is easy to see that a parity condition $\{F_1, F_2, \ldots, F_{2k}\}$ is equivalent to the Streett condition $\{\langle F_{2k-1}, F_{2k-2}\rangle, \ldots, \langle F_3, F_2\rangle, \langle F_1, \emptyset\rangle\}$ and to the Rabin condition $\{\langle F_{2k}, F_{2k-1}\rangle, \ldots, \langle F_4, F_3\rangle, \langle F_2, F_1\rangle\}$. Similarly, a Rabin condition with a single pair $\langle G, B\rangle$, is equivalent to the parity condition $\{B, B \cup G, Q, Q\}$. Generalizing, it is not hard to see that a Rabin condition $\alpha = \{\langle G_1, B_1\rangle, \ldots, \langle G_k, B_k\rangle\}$ with nested "bad" sets, i.e. $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k$, is equivalent to the parity condition $\gamma = \{B_1, B_1 \cup G_1, B_1 \cup G_1 \cup B_2, B_1 \cup G_1 \cup B_2 \cup G_2, \ldots\}$.

"Typed" Translations. "Typed" translations depend on the structure of the associated pre-automaton. For instance, according to [11], deterministic Rabin automata are Büchi type. That is, given a pre-automaton $\mathcal{A}$ and a Rabin condition α, whenever $\mathcal{L}(\mathcal{A}, \alpha)$ is Büchi recognizable there exists a Büchi condition $\beta \subseteq Q$, which is equivalent to the parity condition $\{\emptyset, \beta, Q, Q\}$, such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$. However, the computation of β requires an examination of $\mathcal{A}$ and does not depend only on the syntax of α. As another example, the nested "bad" sets condition above can be relaxed to reproduce another typed translation between Rabin and parity conditions. Indeed, if we replace the $B_i \subseteq B_{i+1}$ condition by a weaker one, namely that for every cycle w in $G_{\mathcal{A}}$, it holds that $w \cap B_i \neq \emptyset \Rightarrow w \cap B_{i+1} \neq \emptyset$, then the translation is still valid, does not change the structure of the automaton, but depends on it. Our goal is to extend the applicability of typed translations to DPWs.

3 The Deterministic Case

In this section we show that a DRW has an equivalent DPW on the same pre-automaton iff it has an equivalent DSW on it. Obviously, the dual result holds when starting from a DSW. Our proof is constructive, providing a polynomial procedure for generating the equivalent parity condition or returning the answer that such a condition does not exist.

Our proof is iterative, proceeding by induction on the index of the generated parity automaton. The first iteration is simple – all the states that cannot be visited infinitely often in a Rabin accepting run (defined below as the "hopeless" states) are gathered to the first (odd) set of the parity condition. After each iteration, the states gathered so far are removed from the pre-automaton, which is then decomposed into maximally strongly connected components. The next iterations are done separately for each component. The second iteration looks for a Rabin pair of the form $\langle G_i, \emptyset\rangle$ (having no "bad" states). If such a pair exists, then its "ultimately good" states are gathered to the next (even) set of the parity condition. The procedure continues, gathering the hopeless states in odd iterations and the ultimately good states in even iterations. In the end, the parity conditions for the separated components are composed to a global condition. The main observation is that an equivalent Streett condition guarantees the existence of the required $\langle G_i, \emptyset\rangle$ pair in every iteration.

We start with defining the notion of "hopeless states". Consider a pre-automaton $\mathcal{A}$ and a Rabin condition α over Q. A state q of $\mathcal{A}$ is *hopeless in $\mathcal{A}$ with respect to α* iff every run r of $\mathcal{A}$ that visits q infinitely often is rejecting. Thus, for every run r of $\mathcal{A}$, if $q \in \mathit{inf}(r)$ then r is rejecting with respect to α. Let $\mathcal{H}_{\mathcal{A},\alpha}$ denote the set of all the states that are hopeless in $\mathcal{A}$ with respect to α. We say that a pre-automaton $\mathcal{A}$ is *hopeless-free* with respect to a condition α iff $\mathcal{H}_{\mathcal{A},\alpha} = \emptyset$.

Note that once a state is hopeless with respect to some condition, it is hopeless for all other equivalent conditions on the same deterministic pre-automaton. We formalize this in the following lemma.

▶ **Lemma 1.** *Let $\mathcal{A}$ be a deterministic pre-automaton and α and β two acceptance conditions over Q such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$. Then, $\mathcal{H}_{\mathcal{A},\alpha} = \mathcal{H}_{\mathcal{A},\beta}$.*

Proof. Once we show that $\mathcal{H}_{\mathcal{A},\alpha} \subseteq \mathcal{H}_{\mathcal{A},\beta}$ the lemma will follow from symmetry. If both $\mathcal{H}_{\mathcal{A},\alpha}$ and $\mathcal{H}_{\mathcal{A},\beta}$ are empty then they are clearly equal. Otherwise, assume w.l.o.g. that $\mathcal{H}_{\mathcal{A},\alpha} \neq \emptyset$ and consider a state $q \in \mathcal{H}_{\mathcal{A},\alpha}$. If no run of $\mathcal{A}$ visits q infinitely often, then $q \in \mathcal{H}_{\mathcal{A},\beta}$ vacuously. Otherwise, consider a run r of $\mathcal{A}$ such that $q \in \mathit{inf}(r)$. Consider a word w over Σ such that r is a run of $\mathcal{A}$ on w. Since $q \in \mathcal{H}_{\mathcal{A},\alpha}$ r is rejecting with respect to α and $w \notin \mathcal{L}(\mathcal{A}, \alpha)$. Since $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$ we have $w \notin \mathcal{L}(\mathcal{A}, \beta)$. Since $\mathcal{A}$ is deterministic r is the single run of $\mathcal{A}$ on w, therefore r is also rejecting with respect to β.

We therefore showed that for every r of $\mathcal{A}$ if $q \in \mathit{inf}(r)$ then r is rejecting with respect to β, therefore $q \in \mathcal{H}_{\mathcal{A},\beta}$. ◀

The next lemmas justify our decomposition and re-composition steps, showing that the equivalence of acceptance conditions is carried over to and from hopeless-free MSCCs.

▶ **Lemma 2** (Zoom In). *Let $\mathcal{A} = \langle \Sigma, Q, \delta, Q_{in} \rangle$ be a deterministic pre-automaton and let α and β be two prefix-independent acceptance conditions such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$. Then, for every $C \subseteq Q$ and reachable state $q \in C$ we have $\mathcal{L}(\mathcal{A}|^q_C, \alpha|_C) = \mathcal{L}(\mathcal{A}|^q_C, \beta|_C)$.*

Proof. From symmetry it suffices to show that $\mathcal{L}(\mathcal{A}|^q_C, \alpha|_C) \subseteq \mathcal{L}(\mathcal{A}|^q_C, \beta|_C)$. Consider a word $w \in \mathcal{L}(\mathcal{A}|^q_C, \alpha|_C)$, and let r be the run of $\mathcal{A}|^q_C$ on w (r is well defined since $\mathcal{A}$ is deterministic, thus so is $\mathcal{A}|^q_C$). Since r is accepting with respect to $\alpha|_C$, the set $\mathit{inf}(r)$ satisfies the condition $\alpha|_C$.

Since $\mathit{inf}(r) \subseteq C$, for every set $S \subseteq Q$ we have that $\mathit{inf}(r) \cap (S \cap C) = \emptyset$ iff $\mathit{inf}(r) \cap S = \emptyset$. Since $\alpha|_C$ is obtained from α by intersecting its sets with C, it follows that $\mathit{inf}(r)$ satisfies also α, hence $w \in \mathcal{L}(\mathcal{A}^q, \alpha)$.

Consider a word $v \in \Sigma^*$ such that $\delta(v) = q$. Such a word clearly exists because q is reachable. Let r' be the run of $\mathcal{A}$ on $v \cdot w$ (the concatenation of the two words). Since $\mathit{inf}(r') = \mathit{inf}(r)$ we have that r' is an accepting run of $\mathcal{A}$ with respect to α, thus $v \cdot w \in \mathcal{L}(\mathcal{A}, \alpha)$. Therefore, we have $v \cdot w \in \mathcal{L}(\mathcal{A}, \beta)$, and since $\mathcal{A}$ is deterministic it follows that r' is accepting with respect to β. Again, since $\mathit{inf}(r') = \mathit{inf}(r)$, we get that r is an accepting run of $\mathcal{A}^q$ with respect to β, and since $\mathit{inf}(r) \subseteq C$, we get that r is an accepting run of $\mathcal{A}|^q_C$ with respect to $\beta|_C$. Thus, $w \in \mathcal{L}(\mathcal{A}|^q_C, \beta|_C)$ as required. ◀

▶ **Lemma 3** (Zoom Out). *Let $\langle \mathcal{A}, \alpha \rangle$ be a deterministic hopeless-free Rabin automaton. Let $\mathcal{C}$ be the set of MSCCs of $G_{\mathcal{A}}$. For every MSCC $C \in \mathcal{C}$, let γ_C be a parity condition of index m_C such that for every state $q \in C$, it holds that $\mathcal{L}(\mathcal{A}|^q_C, \alpha|_C) = \mathcal{L}(\mathcal{A}|^q_C, \gamma_C)$. Then, there is a parity condition γ of index $\max_{C \in \mathcal{C}} m_C$ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$.*

Proof. For every $C \in \mathcal{C}$, let $\gamma_C = \{F_{C,1}, F_{C,2}, \ldots, F_{C,m_C}\}$, and let $m_{\mathcal{C}} = \max_{C \in \mathcal{C}} m_C$. We extend all γ_C to be of index $m_{\mathcal{C}}$ (this can be done by padding the condition with C sets). We define $\gamma = \{F_1, F_2, \ldots, F_{m_{\mathcal{C}}}\}$, where for all $1 \leqslant i \leqslant m_{\mathcal{C}}$, we have $F_i = \bigcup_{C \in \mathcal{C}} F_{C,i}$. We prove that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$.

We first prove that $\mathcal{L}(\mathcal{A}, \alpha) \subseteq \mathcal{L}(\mathcal{A}, \gamma)$. Consider a word $w \in \mathcal{L}(\mathcal{A}, \alpha)$. Let r be the run of $\mathcal{A}$ on w. Then, $\mathit{inf}(r)$ satisfies α. Also, since r is an infinite path in $G_{\mathcal{A}}$ there is a single

$C \in \mathcal{C}$ such that $inf(r) \subseteq C$, thus in particular $inf(r) \cap F_{C',i} = \emptyset$ for every $C' \in \mathcal{C}$ other than C and for every $1 \leqslant i \leqslant m_{\mathcal{C}}$. Thus, the minimal index i for which $inf(r) \cap F_i \neq \emptyset$ is equal to the minimal index i' for which $inf(r) \cap F_{C,i'} \neq \emptyset$.

To see that i' is even, note that there exists an index $l \geqslant 0$ such that $r^l \subseteq C$ and denote $r_l = q$. Because $inf(r) = inf(r^l)$ we have that $w^l \in \mathcal{L}(\mathcal{A}|^q_C, \alpha|_C)$ and therefore also $w^l \in \mathcal{L}(\mathcal{A}|^q_C, \gamma_C)$, thus i' is even.

The other direction is similar. ◀

In fact, Lemma 3 above is valid also for automata that are not hopeless-free. To see this, note that it would be enough to define the sets in γ as $F_i = \mathcal{H}_{\mathcal{A},\alpha} \cup \bigcup_{C \in \mathcal{C}} F_{C,i}$, thus forbidding any accepting run that would be accepting according to γ from visiting hopeless states infinitely often, and therefore restricting the accepting runs to the same MSCCs. We therefore have the following:

▶ **Lemma 4** (Zoom Out). *Let $\langle \mathcal{A}, \alpha \rangle$ be a deterministic Rabin automaton. Let $\mathcal{C}$ be the set of MSCCs of $G_{\mathcal{A}'}$, where $\mathcal{A}'$ is the restriction of $\mathcal{A}$ to its non-hopeless states. For every MSCC $C \in \mathcal{C}$, let γ_C be a parity condition of index m_C such that for every state $q \in C$, it holds that $\mathcal{L}(\mathcal{A}|^q_C, \alpha|_C) = \mathcal{L}(\mathcal{A}|^q_C, \gamma_C)$. Then, there is a parity condition γ of index $\max_{C \in \mathcal{C}} m_C$ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$.*

The Streett Limitation. Lemmas 2 and 4 suggest that we restrict our attention to deterministic strongly connected Rabin and Streett automata that are hopeless-free. The Lemma below provides the key observation that under these conditions one of the “bad” Rabin sets must be empty.

▶ **Lemma 5.** *Let $\mathcal{A}$ be a strongly connected deterministic pre-automaton, and let $\alpha = \{\langle G_1, B_1 \rangle, \ldots, \langle G_k, B_k \rangle\}$ and $\beta = \{\langle L_1, U_1 \rangle, \ldots, \langle L_l, U_l \rangle\}$ be Rabin and Streett conditions such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$. Further assume that $\mathcal{A}$ is hopeless-free with respect to the equivalent conditions α and β. Then, there must be an index $1 \leqslant i \leqslant k$ for which $B_i = \emptyset$.*

Proof. Consider first the Streett condition β. Assume that there is an index $1 \leqslant j \leqslant l$ such that $U_j = \emptyset$. Then, all the states in L_j are hopeless with respect to β. Since $\mathcal{A}$ is hopeless-free with respect to β it follows that if $U_j = \emptyset$ then $L_j = \emptyset$, thus the pair $\langle U_j, L_j \rangle$ can be removed from β. Therefore we can assume that for all $1 \leqslant j \leqslant l$, the set U_j is not empty. Now, consider the Rabin condition α and assume by way of contradiction that for all $1 \leqslant i \leqslant k$ we have $B_i \neq \emptyset$. Consider a word w such that the run r over w visits infinitely often all the states of $\mathcal{A}$. Such a word clearly exists, because $\mathcal{A}$ is strongly connected. Since r visits all the sets B_i of α infinitely often it does not satisfy α. Thus $w \notin \mathcal{L}(\mathcal{A}, \alpha)$. On the other hand, since r visits all the sets U_j (non of which is empty) infinitely often, it does satisfy β, thus $w \in \mathcal{L}(\mathcal{A}, \beta)$, contradicting the equivalence of $\mathcal{L}(\mathcal{A}, \alpha)$ and $\mathcal{L}(\mathcal{A}, \beta)$ ◀

We continue to the main lemma, proving a special case of the desired theorem.

▶ **Lemma 6.** *Let $\mathcal{A}$ be strongly connected deterministic pre-automaton, and let $\alpha = \{\langle G_1, B_1 \rangle, \ldots, \langle G_k, B_k \rangle\}$ and $\beta = \{\langle L_1, U_1 \rangle, \ldots, \langle L_l, U_l \rangle\}$ be Rabin and Streett conditions such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$. Further assume that $\mathcal{A}$ is hopeless-free with respect to the equivalent conditions α and β. Then, there is a parity acceptance condition γ of index at most $2k + 2$ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta) = \mathcal{L}(\mathcal{A}, \gamma)$.*

Proof. The proof proceeds by induction on the index of the Rabin condition. When $k = 1$ it is easy, as $\alpha = \langle G, B \rangle$ is equivalent to the parity condition $\{B, B \cup G, Q, Q\}$. We assume by induction that the claim holds for Rabin conditions of index at most $k-1$. Formally, we assume that given a strongly connected deterministic pre-automaton $\mathcal{A}'$, Rabin and Streett conditions $\alpha = \{\langle G_1, B_1\rangle, \ldots, \langle G_{k-1}, B_{k-1}\rangle\}$ and $\beta = \{\langle L_1, U_1\rangle, \ldots, \langle L_{l'}, U_{l'}\rangle\}$ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$ and such that $\mathcal{A}'$ is hopeless-free with respect to them, we know how to construct a parity acceptance condition γ of index at most $2k$, such that $\mathcal{L}(\mathcal{A}, \alpha') = \mathcal{L}(\mathcal{A}, \beta') = \mathcal{L}(\mathcal{A}, \gamma')$.

We consider a Rabin condition α of index k and decompose it into two Rabin conditions, α' and α'', such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \alpha') \cup \mathcal{L}(\mathcal{A}, \alpha'')$. Next, using the induction hypothesis we will construct parity conditions γ' and γ'' such that $\mathcal{L}(\mathcal{A}', \alpha') = \mathcal{L}(\mathcal{A}', \gamma')$ and $\mathcal{L}(\mathcal{A}', \alpha'') = \mathcal{L}(\mathcal{A}', \gamma'')$. Finally, we will compose γ' and γ'' to get γ.

We start by constructing γ''. According to Lemma 5 (w.l.o.g.) $B_k = \emptyset$. Consider the Rabin condition $\alpha'' = \{\langle G_k, \emptyset\rangle\}$. It is easy to see that α'' is equivalent to the parity condition $\gamma'' = \{\emptyset, G_k, Q, Q\}$. Intuitively, α'' and γ'' accept exactly all the words that could have been accepted thanks to the pair $\langle G_k, B_k\rangle$.

We now proceed to construct γ'. Let $\alpha' = \{\langle G_1, B_1 \cup G_k\rangle, \langle G_2, B_2 \cup G_k\rangle, \ldots, \langle G_{k-1}, B_{k-1} \cup G_k\rangle\}$. Intuitively, α' completes α'' by accepting all the words in $\mathcal{L}(\mathcal{A}, \alpha) \setminus \mathcal{L}(\mathcal{A}, \alpha'')$. It is easy to see that $\mathcal{L}(\mathcal{A}, \alpha') \cap \mathcal{L}(\mathcal{A}, \alpha'') = \emptyset$ and that $\mathcal{L}(\mathcal{A}, \alpha') = \mathcal{L}(\mathcal{A}, \alpha) \setminus \mathcal{L}(\mathcal{A}, \alpha'')$. Consider the Streett condition $\beta' = \{\langle L_1, U_1\rangle, \ldots, \langle L_l, U_l\rangle, \langle G_k, \emptyset\rangle\}$. We claim that β' is equivalent to α' on $\mathcal{A}$. That is, $\mathcal{L}(\mathcal{A}, \beta') = \mathcal{L}(\mathcal{A}, \alpha) \setminus \mathcal{L}(\mathcal{A}, \alpha'')$. To see that $\mathcal{L}(\mathcal{A}, \beta') \subseteq \mathcal{L}(\mathcal{A}, \alpha) \setminus \mathcal{L}(\mathcal{A}, \alpha'')$, consider a word $w \in \Sigma^\omega$, the single run r of $\mathcal{A}$ on w, and the set $\mathit{inf}(r)$. If $\mathit{inf}(r)$ satisfies β' than it clearly satisfies β and therefore $w \in \mathcal{L}(\mathcal{A}, \beta) = \mathcal{L}(\mathcal{A}, \alpha)$. Additionally, r satisfies the Streett pair $\langle G_k, \emptyset\rangle$ which implies that $\mathit{inf}(r) \cap G_k = \emptyset$, so r does not satisfy α' and therefore $w \notin \mathcal{L}(\mathcal{A}, \alpha')$. Showing the inclusion in the other way is similar.

Consider the pre-automaton $\mathcal{A}$ and the acceptance conditions α' and β'. The underlying graph of $\mathcal{A}$ is strongly connected and α' and β' are Rabin and Streett conditions such that $\mathcal{L}(\mathcal{A}, \alpha') = \mathcal{L}(\mathcal{A}, \beta')$ and the index of α' is $k-1$. In order, however, to apply the induction hypothesis, we also need $\mathcal{A}$ to be hopeless-free with respect to α' and β', which is clearly not the case, as the vertices in G_k are hopeless in $\mathcal{A}$ with respect to α'.

Let $\mathcal{C}$ denote the set of MSCCs of $G_{\mathcal{A}'}$, where $\mathcal{A}' = \mathcal{A}_{|Q'}$ and $Q' = Q \setminus \mathcal{H}_{\mathcal{A},\alpha'}$. According to Lemma 2, for every $C \in \mathcal{C}$ and for every $q \in C$ we have $\mathcal{L}(\mathcal{A}|^q_C, \alpha') = \mathcal{L}(\mathcal{A}|^q_C, \beta')$. Each $C \in \mathcal{C}$ is strongly connected and hopeless-free with respect to α'. Hence, the induction hypothesis implies that for each $C \in \mathcal{C}$ there is a parity condition γ^C of index at most $2k$, such that for every $q \in C$, we have $\mathcal{L}(\mathcal{A}|^q_C, \alpha') = \mathcal{L}(\mathcal{A}|^q_C, \gamma^C)$. According to Lemma 4, this implies the existence of a single condition $\gamma' = \{F'_1, \ldots, F'_{2k}\}$ such that $\mathcal{L}(\mathcal{A}, \alpha') = \mathcal{L}(\mathcal{A}, \gamma')$.

We define γ as the composition of γ' and γ''. Formally, $\gamma = \{\emptyset, G_k, F_3, \ldots, F_{2k+2}\}$ where for all $3 \leqslant i \leqslant m+2$, we set $F_i = F'_{i-2}$.

We show that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$. Consider a word $w \in \Sigma^\omega$, the single run r of $\mathcal{A}$ on w, and the set $\mathit{inf}(r)$. If $w \in \mathcal{L}(\mathcal{A}, \alpha)$ then r satisfies α and there is an index $1 \leqslant i \leqslant k$ such that $\mathit{inf}(r) \cap G_i \neq \emptyset$ and $\mathit{inf}(r) \cap B_i = \emptyset$. If $\mathit{inf}(r) \cap G_k \neq \emptyset$ (i.e. $i = k$) then the minimal index for which $\mathit{inf}(r) \cap F_j \neq \emptyset$ is 2, which is even, and therefore r satisfies γ. Otherwise, r does not satisfy α'' and therefore since it is accepting and since $\mathcal{L}(\mathcal{A}, \alpha') = \mathcal{L}(\mathcal{A}, \alpha) \setminus \mathcal{L}(\mathcal{A}, \alpha'')$ it must satisfy α'. This, in turn, implies that r also satisfies γ'. Let j be the minimal index for which $\mathit{inf}(r) \cap F'_i \neq \emptyset$. Since r satisfies γ' the index j is even. Clearly, the minimal index for which $\mathit{inf}(r) \cap F_i \neq \emptyset$ equals $j+2$, and is therefore also even, and therefore r satisfies γ, thus $w \in \mathcal{L}(\mathcal{A}, \gamma)$ and we have $\mathcal{L}(\mathcal{A}, \alpha) \subseteq \mathcal{L}(\mathcal{A}, \gamma)$.

The other direction of the inclusion is similar. ◀

Lemmas 2 and 4 imply the generalization of Lemma 6 to any pre-automaton. Formally, we have the following.

▶ **Theorem 7.** *Let $\mathcal{A}$ be a deterministic pre-automaton with n states, α a Rabin condition of index k and β a Streett condition of index l such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$. Then, there exists a parity condition γ of index at most $\min\{2k+2, 2l+2, n+2\}$ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta) = \mathcal{L}(\mathcal{A}, \gamma)$.*

Proof. The proof follows a similar argumentation to the induction step in Lemma 6. Let $\mathcal{C}$ denote the set of MSCCs of $G_{\mathcal{A}'}$, where $\mathcal{A}' = \mathcal{A}_{|Q'}$ and $Q' = Q \setminus \mathcal{H}_{\mathcal{A},\alpha}$. According to Lemma 2, for every $C \in \mathcal{C}$ and for every $q \in C$ we have $\mathcal{L}(\mathcal{A}|_C^q, \alpha) = \mathcal{L}(\mathcal{A}|_C^q, \beta)$. Each $C \in \mathcal{C}$ is strongly connected and hopeless-free with respect to α. Hence, Lemma 6 implies that for each $C \in \mathcal{C}$ there is a parity condition γ^C of index at most $2k+2$, such that for every $q \in C$, we have $\mathcal{L}(\mathcal{A}|_C^q, \alpha') = \mathcal{L}(\mathcal{A}|_C^q, \gamma^C)$. According to Lemma 4, this implies the existence of a single condition $\gamma = \{F_1, \ldots, F_m\}$ of index at most $2k+2$ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$.

By the above proof, the index of γ is at most $2k+2$. We now tighten it further. If $k > l$ we can switch the roles of α and β: Let $\tilde{\alpha}$ and $\tilde{\beta}$ be α and β when viewed as Streett and Rabin conditions, respectively. We still have $\mathcal{L}(\mathcal{A}, \tilde{\alpha}) = \mathcal{L}(\mathcal{A}, \tilde{\beta})$, recognizing the complementing language. By the above, there is a parity condition $\tilde{\gamma}$ of degree at most $2l+2$ such that $\mathcal{L}(\mathcal{A}, \tilde{\gamma}) = \mathcal{L}(\mathcal{A}, \tilde{\beta})$. In order to obtain a parity condition that would be equivalent to the original α, we dualize $\tilde{\gamma}$ (the way we have constructed $\tilde{\gamma}$ guarantees that a dualization would not involve an increase in the index). Finally, a parity condition on n states that has more than $n+2$ sets must contain equivalent subsequent sets and can therefore be simplified to one with at most $n+2$ sets. Hence the $\min\{2k+2, 2l+2, n+2\}$ bound. ◀

As discussed in Section 1, the considerations behind our proof are different than these used in [22] in the context of two-player games. In addition, our proof is constructive, and it generates the equivalent parity condition. It is not clear to us whether and how the proof in [22] can be adopted to the setting of automata. In particular, an attempt to generate a parity condition following the considerations in [22] involves an examination of subsets of the state space of the game, and is thus exponential. As we show below, our procedure requires only polynomial time.

PTIME-completeness. The proof above is constructive, allowing to generate the equivalent parity condition or return the answer that such a condition does not exist. We show below that our procedure is in PTIME and that the related question is indeed PTIME-complete.

▶ **Theorem 8.** *Consider a DRW or a DSW $\mathcal{A}$. The problem of deciding whether $\mathcal{A}$ has an equivalent DPW on the same structure is PTIME-complete.*

Proof. We prove the result for DRW. By the duality of the Rabin and Streett conditions, and the self-duality of the parity condition, the result for DSW follows.

We start with the upper bound. Assume that $\mathcal{A}$ has n states and its acceptance condition α is of index k. As discussed above, the given Streett condition does not play a role in the construction of the parity condition, and its essence is in guaranteeing the existence of a Rabin pair with an empty "bad" set. Accordingly, the procedure described above for generating γ works for every DRW and it always ends after up to $\min(n+1, k)$ iterations. Each iteration is clearly in PTIME, as it only marks the hopeless states, which can be done by exploring the loops in the automaton's graph. If it completes all iterations, then the DRW has an equivalent DPW on the same structure (the one generated by the procedure). Otherwise, the procedure gets stuck in an iteration in which no Rabin pair with an empty bad set exists, in which case the DRW does not have an equivalent DPW on the same structure.

It is left to prove PTIME-hardness. We do a reduction from DRW universality, which is dual to DSW emptiness, proved to be PTIME-complete in [6]. In the proof, we consider languages over an alphabet $\Sigma_1 \times \Sigma_2$. For a word $w \in (\Sigma_1 \times \Sigma_2)^\omega$, let $w_1 \in \Sigma_1^\omega$ be the word obtained from w by projecting its letters on Σ_1, and similarly for w_2 and Σ_2. For words $x_1 \in \Sigma_1^\omega$ and $x_2 \in {\Sigma_2}^\omega$, let $x_1 \oplus x_2$ denote the word $w \in (\Sigma_1 \times \Sigma_2)^\omega$ with $w_1 = x_1$ and $w_2 = x_2$. Given a DRW $\mathcal{R}$ with alphabet Σ_1, we define another DRW $\mathcal{A}$ such that $\mathcal{R}$ is universal (that is, $L(\mathcal{R}) = \Sigma_1^\omega$) iff $\mathcal{A}$ has an equivalent DPW on the same structure. Let $\mathcal{R} = \langle \Sigma_1, Q, q_0, \delta, \alpha \rangle$, and let $\mathcal{R}' = \langle \Sigma_2, Q', q'_0, \delta', \alpha' \rangle$ be a DRW such that there exists no DPW equivalent to $\mathcal{R}'$ on the same structure. We define $\mathcal{A} = \langle \Sigma_1 \times \Sigma_2, Q \times Q', \langle q_0, q'_0 \rangle, \delta'', \alpha'' \rangle$, where

- $\delta''(\langle q, q' \rangle, \langle \sigma_1, \sigma_2 \rangle) = \langle \delta(q, \sigma_1), \delta'(q', \sigma_2) \rangle$, and
- $\alpha'' = \{\langle G \times Q', B \times Q' \rangle : \langle G, B \rangle \in \alpha\} \cup \{\langle Q \times G', Q \times B' \rangle : \langle G', B' \rangle \in \alpha'\}$.

It is easy to see that $\mathcal{L}(\mathcal{A}) = \{w \ : w_1 \in \mathcal{L}(\mathcal{R}) \text{ or } w_2 \in \mathcal{L}(\mathcal{R}')\}$. We prove that $\mathcal{R}$ is universal iff $\mathcal{A}$ has an equivalent DPW on the same structure. First, if $\mathcal{R}$ is universal, so is $\mathcal{A}$, and hence it clearly has an equivalent DPW on the same structure. Assume now that $\mathcal{R}$ is not universal, we show that there is no DPW equivalent to $\mathcal{A}$ on the same structure. Assume by way of contradiction that γ is a parity condition defined on top of $Q \times Q'$ such that the language of $\mathcal{A}$ with γ is equal to $\mathcal{L}(\mathcal{A})$. In the full version we prove that the projection of γ on Q' results in a parity condition γ' such that the language of $\mathcal{R}'$ with acceptance condition γ' is equivalent to $\mathcal{L}(\mathcal{R}')$. This, however, contradicts the assumption that no DPW equivalent to $\mathcal{R}'$ can be defined on the same structure. Essentially, the claim follows from the fact that if $w_1 \in \Sigma_1^\omega$ is a word rejected by $\mathcal{R}$ (since $\mathcal{R}$ is not universal, such a word exists), then the behavior of $\mathcal{A}$ on words whose projection on Σ_1 is w_1 depends only on its $\mathcal{R}'$ component. ◀

4 The Non-Deterministic Case

Can our results be generalized to the nondeterministic case? To show the converse we describe a nondeterministic pre-automaton $\mathcal{A}$ on top of which we define Rabin and Streett conditions, α and β, such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \beta)$, however there is no parity condition γ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$. It follows that our main result does not hold in the nondeterministic setting. Furthermore, by dualizing one gets a counterexample for the claim about universal automata (that is, alternating automata in which transitions are only conjunctively related). Indeed, the key role of the determinism in our proof is inevitable . We prove that the problem of deciding whether a given NRW or NSW has an equivalent NPW on the same structure is PSPACE-complete.

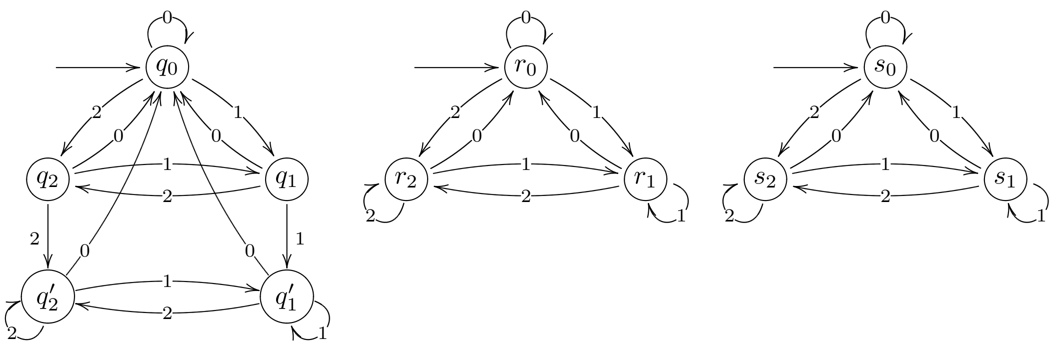

Figure 1 A nondeterministic pre-automaton $\mathcal{A}$ having equivalent Rabin and Streett conditions for L with no corresponding parity condition.

Consider the nondeterministic pre-automaton $\mathcal{A}$ over $\Sigma = \{0, 1, 2\}$ depicted in Figure 1. We use Q, R and S to denote the sets of states of the different components, thus $Q = \{q_0, q_1, q'_1, q_2, q'_2\}, R = \{r_0, r_1, r_2\}$ and $S = \{s_0, s_1, s_2\}$. Note that the nondeterminism of $\mathcal{A}$ is limited to the choice of initial state , thus it is a non-ambiguous (or a single-run) automaton. Consider the Rabin and Streett conditions

- $\alpha = \{\langle\{q_1\}, \{q_2, q_2'\}\rangle, \langle\{q_2\}, \{q_1, q_1'\}\rangle\}$ (Rabin), and
- $\beta = \{\langle Q \cup \{r_1, s_2\}, \emptyset\rangle, \langle R, \{r_0\}\rangle, \langle R, \{r_2\}\rangle, \langle S, \{s_0\}\rangle, \langle S, \{s_1\}\rangle\}$ (Streett).

We define $\mathcal{L} = \{w \in \Sigma^\omega : \mathit{inf}(w) = \{0, 1\} \text{ or } \mathit{inf}(w) = \{0, 2\}\}$ and show that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L} = \mathcal{L}(\mathcal{A}, \beta)$. It is easy to see that a word $w \in \mathcal{L}$ has an accepting run. In fact, the single run of $\mathcal{A}$ on w that is accepting with respect to α is the one that starts at q_0. On the other hand, a word $w \in \Sigma^\omega$ belongs to $\mathcal{L}(\mathcal{A}, \alpha)$ if there is a run r of $\mathcal{A}$ that satisfies α. Such a run must start at q_0, as otherwise r never visits any of α's "good" sets. Further, to satisfy the pair $\langle\{q_1\}, \{q_2, q_2'\}\rangle$ the run r must get trapped in the right part of Q, so w must contain only finitely many 2's and infinitely many 0's and 1's. Similarly, in order to satisfy the pair $\langle\{q_2\}, \{q_1, q_1'\}\rangle$, the run r must get trapped in the left part of Q, so it must consist of only finitely many 1's and infinitely many 0's and 2's.

Consider $\mathcal{L}(\mathcal{A}, \beta)$. It is easy to see that a word $w \in \mathcal{L}$ has an accepting run. In fact, if w has only finitely many 1's then the single run of $\mathcal{A}$ on w that is accepting with respect to β is the one that starts at r_0, and if w has only finitely many 2's then the single run of $\mathcal{A}$ on w that is accepting with respect to β is the one that starts at s_0. On the other hand, a word $w \in \Sigma^\omega$ belongs to $\mathcal{L}(\mathcal{A}, \beta)$ if there is a run r of $\mathcal{A}$ that satisfies β. Such a run must either start at r_0 or at s_0, as otherwise it will be trapped in Q and violate the pair $\langle Q \cup \{r_1, s_2\}, \emptyset\rangle$. If r starts at the state r_0, then in order to accept, w must consist of infinitely many 0's and 2's but only finitely many 1's. Otherwise, r must start at s_0, then in order to accept, w must consist of infinitely many 0's and 1's but only finitely many 2's.

We now show that there is no parity condition γ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$. Assume by contradiction that $\gamma = \{F_1, F_2, \ldots, F_m\}$ such that $\mathcal{L}(\mathcal{A}, \alpha) = \mathcal{L}(\mathcal{A}, \gamma)$. Referring to the minimal index i for which $q \in F_i$ by the *rank* or q, we note that all states with self loops cannot have an even rank, as otherwise words that consist of a single letter can have an accepting run. Hence if a run is accepting with respect to γ it must be contained in Q. Since $(01)^\omega \in \mathcal{L}(\mathcal{A}, \alpha)$ it must also be in $\mathcal{L}(\mathcal{A}, \gamma)$, therefore the run $(q_0, q_1)^\omega$ must be accepting with respect to γ. Since q_0 has a self loop it cannot be ranked evenly, therefore the rank of q_1 must be even. Similarly, q_2's rank must also be even. However, that would imply that the run $(q_1 q_2)^\omega$ on $(12)^\omega$ would be accepting with respect to γ, but $(12)^\omega \notin \mathcal{L}(\mathcal{A}, \alpha)$.

PSPACE-completeness. The counter example above suggests that the translation to an equivalent parity condition on the same structure is more complicated in the nondeterministic setting. Indeed, we show below that this problem is PSPACE-complete.

▶ **Theorem 9.** *Consider an NRW or an NSW $\mathcal{A}$. The problem of deciding whether $\mathcal{A}$ has an equivalent NPW on the same structure is PSPACE-complete.*

Proof. For the upper bound, one can go over all possible parity conditions for $\mathcal{A}$ and check the equivalence of the obtained NPW with $\mathcal{A}$. The lower bound is similar to the one described in the proof of Theorem 8, only that here the Rabin and Streett cases are not dual (dualizing an NRW, one gets a universal (rather than nondeterministic) Streett automaton), thus we have to consider both cases. In addition, for the lower bounds, while the reductions are still from the universality problem, now they are from NRW or NSW universality, which are PSPACE-complete. ◀

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